ON A CONTACT PROBLEM CONNECTED WITH THE TORSION OF A HOLLOW SEMISPHERE

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The problem of the torsion of a conical shaft, when the shaft is twisted by moments acting on its transverse faces, was first treated by Foppl [1]. The torsion of a shaft in the form of a cone has also been treated in the publications of Lokshin [2] and Panarin [3]. The torsion of a sphere, when it is twisted by concentrated torques, was treated in the papers of Snell [4] and Huber [5]. Some related problems on the torsion of shafts of variable section were treated in the works of Solianik-Krassa [6] and Abramian [7].

Das [8] has treated the torsion of a body of revolution, the axial section of which is bounded by two spherical surfaces and a conical surface, with a conical surface separating the two parts of the body which consist of different materials. The cone is twisted by shear stresses applied to the conical surface and the spherical surfaces are fixed. The solution of this problem was obtained with the aid of conical functions.

Another paper by Das [9] treats the torsion of a composite sphere and that of spheroids by loading distributed over the surface according to some definite law. The torsion of a semispherical shell of two layers, subjected to an arbitrary axisymmetric loading, was treated in [10].

The present paper deals with the problem of the torsion of a semispherical shell, when part of its transverse face is fixed and it is twisted by an arbitrary axisymmetric loading. The solution of the problem is obtained with the aid of conical functions. The determination of the constants of integration has been reduced to the solution of an infinite system of linear equations. It is proved that this system is completely regular and has free terms bounded from above.

1. The problem of the torsion of a body of revolution will be solved with the aid of the displacement function $\psi(r, z)$, which, in the region of the cross-section of the body of revolution, satisfies the differential equation

$$\frac{\partial^2\psi}{\partial r^2}+\frac{3}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial z^2}=0 \tag{1.1}$$

and on the boundary of the transverse section either its normal derivative or the function itself is prescribed.

The stresses $\tau_{r\varphi}$, $\tau_{\varphi z}$ and the displacement v are expressed in terms of the displacement function $\psi(r, z)$ by means of the formulas

$$\tau_{r\varphi}=Gr\frac{\partial\psi}{\partial r}, \qquad \tau_{\varphi z}=Gz\frac{\partial\psi}{\partial z}, \qquad v=r\psi(r,z) \tag{1.2}$$

where G is the shear modulus of the material.

Passing to the new coordinate system

$$r=ae^t\sqrt{1-\xi^2}, \qquad z=ae^t\xi, \qquad \xi=\frac{z}{\sqrt{r^2+z^2}}, \qquad t=\ln\frac{\sqrt{r^2+z^2}}{a} \tag{1.3}$$

where $a > 0$ is a positive constant, Equation (1.1) takes the form

$$\frac{\partial^2\psi}{\partial t^2}+(1-\xi^2)\frac{\partial^2\psi}{\partial\xi^2}+3\frac{\partial\psi}{\partial t}-4\xi\frac{\partial\psi}{\partial\xi}=0 \tag{1.4}$$

The formulas for the stresses and displacements assume the form

$$\tau_{\xi\varphi}=-G(1-\xi^2)\frac{\partial\psi}{\partial\xi}, \qquad \tau_{t\varphi}=G\sqrt{1-\xi^2}\frac{\partial\psi}{\partial t}$$
$$v=ae^t\sqrt{1-\xi^2}\,\psi(t,\xi) \tag{1.5}$$

Solving Equation (1.4) by the method of separation of variables, we obtain for the function $\psi(t, \xi)$ the following particular solutions:

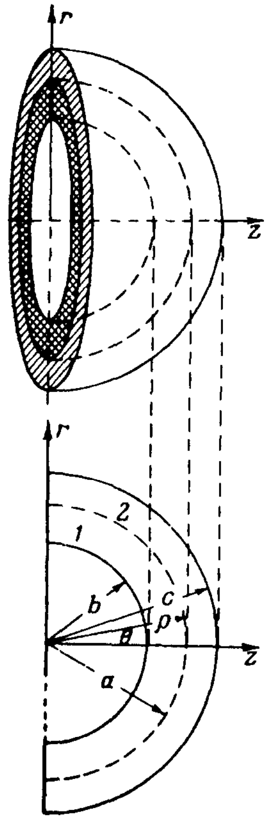
$$\psi(t,\xi)=e^{-3t/2}\left[A\sinh\frac{(2n+1)}{2}t+B\cosh\frac{(2n+1)}{2}t\right]\varphi_n(\xi)$$
$$\psi(t,\xi)=\frac{d}{d\xi}\left[AP_{-1/2+\mu_k i}(\xi)+BQ_{-1/2+\mu_k i}(\xi)\right]T_k(t) \tag{1.6}$$

where

$$\varphi_n(\xi)=\frac{d}{d\xi}[CP_n(\xi)+DQ_n(\xi)]$$
$$T_k(t)=e^{-3t/2}[C\sin\mu_k t+D\cos\mu_k t] \tag{1.7}$$

Here $P_n(\xi)$ and $Q_n(\xi)$ are Legendre functions [11], $P_{-1/2+\mu_k i}(\xi)$ and

$Q_{-1/2+\mu_k i}(\xi)$ are conical functions [11], [12], A, B, C and D are arbitrary constants, and n is a positive natural number.



We observe that the functions

$$1, \quad e^{-3t}, \quad \frac{1}{1\pm\xi} - \ln(1\pm\xi) - t \tag{1.9}$$

are also particular solutions of the differential equation (1.4).

2. We will consider the problem of the torsion of a spherical shell with radii b and c $(b < c)$, when torsional shear stresses act on the spherical surfaces and on the annular part of the plane $z = 0$ $(r > a,\ b < a < c)$ (see figure), and on the remaining part of the plane $z = 0$ $(r < a)$ the displacement v is prescribed.

For such a problem the boundary conditions will have the form

$$\begin{aligned} &\tau_{t\varphi}(-t_1, \xi) = f_1(\xi) \\ &\tau_{t\varphi}(t_2, \xi) = f_2(\xi) \end{aligned} \qquad (0 \leqslant \xi \leqslant 1)$$

$$\begin{aligned} &\tau_{\xi\varphi}(t, 0) = f_3(t) \qquad (0 \leqslant t \leqslant t_2) \\ &v(t, 0) = f_4(t) \qquad (-t_1 \leqslant t < 0) \end{aligned} \tag{2.1}$$

where

$$t_1 = \ln\frac{a}{b}, \qquad t_2 = \ln\frac{c}{a} \tag{2.2}$$

$$(0 \leqslant \xi \leqslant 1)$$

The function $\psi(t, \xi)$ assumes the form

$$\psi(t, \xi) = \begin{cases} \psi_1(t, \xi) \text{ in the region } 1 (t < 0) \\ \psi_2(t, \xi) \text{ in the region } 2 (t > 0) \end{cases} \tag{2.3}$$

From (1.4) and (2.3) it follows that the functions ψ_1 and ψ_2 satisfy Equation (1.4). From (2.1) and (2.3) we obtain the following boundary conditions for the functions ψ_1 and ψ_2:

$$G\sqrt{1-\xi^2}\frac{\partial\psi_1}{\partial t}\bigg|_{t=-t_1} = f_1(\xi), \qquad -G\frac{\partial\psi_2}{\partial\xi}\bigg|_{\xi=0} = f_3(t)$$

$$G\sqrt{1-\xi^2}\frac{\partial\psi_2}{\partial t}\bigg|_{t=t_2} = f_2(\xi), \qquad ae^t\psi_1(t, 0) = f_4(t) \tag{2.4}$$

Moreover, conditions of contiguity must be satisfied on the line of contact between the regions 1 and 2:

$$\psi_1(0,\xi)=\psi_2(0,\xi) \qquad \left.\frac{\partial\psi_1}{\partial t}\right|_{t=0}=\left.\frac{\partial\psi_2}{\partial t}\right|_{t=0} \tag{2.5}$$

The solutions for the functions ψ_1 and ψ_2 will be taken in the form

$$\psi_1(t,\xi)=A^{(1)}+B^{(1)}e^{-3t}+D^{(1)}\left[\frac{1}{1+\xi}-\ln(1+\xi)-t\right]+$$
$$+\sum_{k=1}^{\infty}e^{-3t/2}\left[A_k^{(1)}\sinh\beta_k t+B_k^{(1)}\cosh\beta_k t\right]\varphi_{2k+1}(\xi)+\sum_{k=1}^{\infty}D_k^{(1)}P'_{-1/2+\mu_k i}(\xi)\,T_k^{(1)}(t)$$
$$(-t_1\leqslant t<0,\ 0\leqslant\xi\leqslant 1) \tag{2.6}$$

$$\psi_2(t,\xi)=A^{(2)}+B^{(2)}e^{-3t}+D^{(2)}\left[\frac{1}{1+\xi}-\ln(1+\xi)-t\right]+$$
$$+\sum_{k=1}^{\infty}e^{-3t/2}\left[A_k^{(2)}\sinh\beta_k t+B_k^{(2)}\cosh\beta_k t\right]\varphi_{2k+1}(\xi)+\sum_{k=1}^{\infty}D_k^{(2)}P'_{-1/2+\gamma_k i}(\xi)\,T_k^{(2)}(t)$$
$$(0\leqslant t\leqslant t_2,\ 0\leqslant\xi\leqslant 1) \tag{2.7}$$

where

$$\begin{aligned}T_k^{(1)}(t)&=e^{-3t/2}\left(\tfrac{3}{2}\sin\mu_k t+\mu_k\cos\mu_k t\right)\\ T_k^{(2)}(t)&=e^{-3t/2}\left(\tfrac{3}{2}\sin\gamma_k t+\gamma_k\cos\gamma_k t\right)\end{aligned} \tag{2.8}$$

$$\varphi_{2k+1}(\xi)=\frac{d}{d\xi}P_{2k+1}(\xi) \tag{2.9}$$

$$\begin{gathered}\beta_k=\frac{4k+3}{2}, \qquad \mu_k=\frac{k\pi}{t_1}, \qquad \gamma_k=\frac{k\pi}{t_2} \qquad (k=1,2,3,\ldots)\\ \mu_0=\gamma_0=\tfrac{3}{2}i, \qquad i=\sqrt{-1}\end{gathered} \tag{2.10}$$

3. We will prove that the functions [13]

$$\chi_k(t)=\begin{cases}e^{\alpha t} & \text{when } k=0\\ \alpha\sin\gamma_k t+\gamma_k\cos\gamma_k t & \text{when } k=1,2,3,\ldots\end{cases} \tag{3.1}$$

where α is a constant, form a closed orthogonal system in the interval $[0,\ t_2]$ with respect to functions which satisfy the Dirichlet conditions. For this it is necessary to prove that, from the conditions

$$\int_0^{t_2} f(t)\,\chi_k(t)\,dt = 0 \qquad (k = 0, 1, 2, \ldots) \tag{3.2}$$

where $f(t)$ are arbitrary functions in the class $L^2[0,\ t_2]$, it follows that $f(t) \equiv 0$ or, what amounts to exactly the same thing, that from the conditions

$$\int_0^{t_2} f(t)\,\chi_k(t)\,dt = 0 \qquad (k = 1, 2, 3, \ldots) \tag{3.3}$$

it follows that the function $f(t)$ either corresponds with the function $\chi_0(t) = e^{\alpha t}$ or is identically zero.

We substitute the value of $\chi_k(t)$ from (3.1) into (3.3) and transform the resulting integral

$$\int_0^{t_2} f(t)\,\chi_k(t)\,dt = \alpha \int_0^{t_2} f(t) \sin \gamma_k\, t dt + \gamma_k \int_0^{t_2} f(t) \cos \gamma_k\, t\, dt =$$

$$= \int_0^{t_2} [\alpha f(t) - f'(t)] \sin \gamma_k\, t\, dt$$

Then the conditions (3.3) assume the form

$$\int_0^{t_2} [\alpha f(t) - f'(t)] \sin \gamma_k\, t\, dt = 0 \qquad (k = 1, 2, 3, \ldots) \tag{3.4}$$

Since the system of functions $\{\sin \gamma_k t\}$ $(k = 1,\ 2,\ 3,\ \ldots)$ is complete in $L^2[0,\ t_2]$, it follows from (3.4) that the function $f(t)$ must satisfy the following differential equation

$$\alpha f(t) - f'(t) = 0$$

which has the unique non-vanishing solution

$$f(t) = Ce^{\alpha t} \tag{3.5}$$

i.e. the function $f(t)$ which satisfies the conditions (3.3) is either identically zero or it coincides with the function $\chi_0(t) = e^{\alpha t}$, with an accuracy up to a numerical multiplicative factor.

From the values of the integral

$$\int_0^{t_2} \chi_k(t)\,\chi_p(t)\,dt = \begin{cases} \frac{t_2}{2}(\gamma_k^2+\alpha^2) & \text{when } k=p\neq 0 \\ \frac{1}{2\alpha}(e^{2\alpha t_2}-1) & \text{when } k=p=0 \\ 0 & \text{when } k\neq p \end{cases} \tag{3.6}$$

it follows that the functions $\chi_k(t)$ form an orthogonal system.

Thus the system of functions $\{\chi_k\}$ is complete in $L^2[0,\ t_2]$. Hence it follows that an arbitrary function $f(t)\in L^2[0,\ t_2]$ can be expanded in a Fourier series of the functions (3.1). Moreover, at a point of continuity of the function $f(t)$, we will have

$$f(t) = a_0\chi_0(t) + \sum_{k=1}^{\infty} a_k\,\chi_k(t) \tag{3.7}$$

where, by virtue of (3.6), the coefficients of the expansion are determined by the formulas

$$a_0 = \frac{2\alpha}{e^{2\alpha t_2}-1}\int_0^{t_2} f(t)\,\chi_0(t)\,dt, \qquad a_k = \frac{2}{t_2(\gamma_k^2+\alpha^2)}\int_0^{t_2} f(t)\,\chi_k(t)\,dt \tag{3.8}$$

Thus the systems of functions $\{1,\ T_k^{(1)}(t)\}$ and $\{1,\ T_k^{(2)}(t)\}$ are closed and orthogonal with the weight factor e^{3t} in the intervals $[-t_1,\ 0]$ and $[0,\ t_2]$ respectively, i.e. for these functions we have

$$\int_{-t_1}^{0} e^{3t}\,T_k^{(1)}(t)\,T_p^{(1)}(t)\,dt = \begin{cases} 0 & k\neq p \\ \frac{1}{2}t_1\left(\mu_k^2+\frac{9}{4}\right) & k=p\neq 0 \end{cases} \tag{3.9}$$

$$\int_0^{t_2} e^{3t}\,T_k^{(2)}(t)\,T_p^{(2)}(t)\,dt = \begin{cases} 0 & k\neq p \\ \frac{1}{2}t_2\left(\gamma_k^2+\frac{9}{4}\right) & k=p\neq 0 \end{cases} \tag{3.10}$$

$$\int_{-t_1}^{0} e^{3t}\,T_k^{(1)}(t)\,dt = \int_0^{t_2} e^{3t}\,T_k^{(2)}(t)\,dt = 0 \tag{3.11}$$

The functions $\{1,\ \varphi_{2k+1}(\xi)\}$ are closed and orthogonal with the weight factor $(1-\xi^2)$ in the interval $[0,\ 1]$, i.e. they satisfy the equalities

$$\int_0^1 (1-\xi^2)\,\varphi_{2k+1}(\xi)\,\varphi_{2p+1}(\xi)\,d\xi = \begin{cases} 0 & (k\neq p) \\ \omega_p = \frac{(2p+1)(2p+2)}{4p+3} & (k=p) \end{cases} \tag{3.12}$$

$$\int_0^1 (1-\xi^2)\,\varphi_{2k+1}(\xi)\,d\xi = 0 \tag{3.13}$$

4. For the purpose of satisfying the boundary conditions (2.4) and (2.5), we will represent the arbitrary functions f_i in the form of series

$$
\begin{aligned}
f_1(\xi) &= G\sqrt{1-\xi^2}\,\Big[a_0 + \sum_{k=1}^{\infty} a_k\,\varphi_{2k+1}(\xi)\Big] \\
f_2(\xi) &= G\sqrt{1-\xi^2}\,\Big[b_0 + \sum_{k=1}^{\infty} b_k\,\varphi_{2k+1}(\xi)\Big]
\end{aligned}
\qquad (0 \leqslant \xi \leqslant 1) \tag{4.1}
$$

$$
\begin{aligned}
f_3(t) &= G\Big[c_0 + \sum_{k=1}^{\infty} c_k\,T_k^{(2)}(t)\Big] \qquad && (0 \leqslant t \leqslant t_2) \\
f_4(t) &= ae^t\Big[d_0 + \sum_{k=1}^{\infty} d_k\,T_k^{(1)}(t)\Big] \qquad && (-t_1 \leqslant t < 0)
\end{aligned}
\tag{4.2}
$$

where, in accordance with the relations (3.8) to (3.13), the coefficients of the expansions are determined by the following formulas:

$$
\begin{aligned}
a_0 &= \frac{3}{2G}\int_0^1 \sqrt{1-\xi^2}\,f_1(\xi)\,d\xi, \qquad & a_k &= \frac{1}{G\omega_k}\int_0^1 \sqrt{1-\xi^2}\,f_1(\xi)\,\varphi_{2k+1}(\xi)\,d\xi \\
b_0 &= \frac{3}{2G}\int_0^1 \sqrt{1-\xi^2}\,f_2(\xi)\,d\xi, \qquad & b_k &= \frac{1}{G\omega_k}\int_0^1 \sqrt{1-\xi^2}\,f_2(\xi)\,\varphi_{2k+1}(\xi)\,d\xi
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
c_0 &= \frac{3}{G(e^{3t_2}-1)}\int_0^{t_2} e^{3t} f_3(t)\,dt, \qquad & c_k &= \frac{2}{Gt_2(\gamma_k^2 + {}^9/_4)}\int_0^{t_2} e^{3t} f_3(t)\,T_k^{(2)}(t)\,dt \\
d_0 &= \frac{3}{a(1-e^{-3t_1})}\int_{-t_1}^0 e^{2t} f_4(t)\,dt, \qquad & d_k &= \frac{2}{at_1(\mu_k^2 + {}^9/_4)}\int_{-t_1}^0 e^{2t} f_4(t)\,T_k^{(1)}(t)\,dt
\end{aligned}
\tag{4.4}
$$

In satisfying the boundary conditions and the conditions for contact we obtain a series of relations for the determination of the unknown coefficients. Solving these equations for the unknowns, we obtain the following expressions (m is a constant number to be determined later on)

$$A_k^{(1)} = \frac{1}{\beta_k^2 - {}^9/_4}\left[\frac{mY_k(4k+3)\,\varphi_{2k+1}(0)}{(2k+1)(2k+2)}\left(\beta_k + \frac{3}{2}\coth\beta_k t_1\right) - \frac{3a_k\,e^{-3t_1/2}}{2\sinh\beta_k t_1}\right]$$

$$B_k^{(1)} = \frac{1}{\beta_k^2 - {}^9/_4}\left[\frac{mY_k(4k+3)\,\varphi_{2k+1}(0)}{(2k+1)(2k+2)}\left(\frac{3}{2} + \beta_k\coth\beta_k t_1\right) - \frac{\beta_k a_k\,e^{-3t_1/2}}{\sinh\beta_k t_1}\right]$$

$$A_k^{(2)} = \frac{1}{\beta_k^2 - 9/4}\left[\frac{mY_k(4k+3)\varphi_{2k+1}(0)}{(2k+1)(2k+2)}\left(\beta_k - \frac{3}{2}\coth\beta_k t_2\right) + \frac{3b_k e^{3t_2/2}}{2\sinh\beta_k t_2}\right]$$

$$B_k^{(2)} = \frac{1}{\beta_k^2 - 9/4}\left[\frac{mY_k(4k+3)\varphi_{2k+1}(0)}{(2k+1)(2k+2)}\left(\frac{3}{2} - \beta_k\coth\beta_k t_2\right) + \frac{\beta_k b_k e^{3t_2/2}}{\sinh\beta_k t_2}\right]$$

$$D_k^{(1)} = \frac{X_k + \gamma_k c_k}{\mu_k P''_{-1/2+\mu_k i}(0)}, \qquad D_k^{(2)} = \frac{c_k}{P''_{-1/2+\mu_k i}(0)} \tag{4.5}$$

$$B^{(1)} = \frac{c_0 - (c_0 - 2b_0)e^{3t_2}}{6(e^{3t_1} - 1)}, \qquad B^{(2)} = \frac{c_0 - 2b_0}{6}e^{3t_1}$$

$$D^{(1)} = -a_0 - e^{3t_1}\frac{c_0 - (c_0 - 2b_0)e^{3t_2}}{2(e^{3t_1} - 1)}, \qquad D^{(2)} = -\frac{c_0}{2}$$

$$A^{(1)} = d_0 - D^{(1)} - \frac{3}{1 - e^{-3t_1}}\left[B^{(1)}t_1 + \sum_{k=1}^{\infty}\frac{(4k+3)\varphi_{2k+1}^2(0)\,mY_k}{(2k+1)(2k+2)(\beta_k^2 - 9/4)} - e^{3t_1}\sum_{k=1}^{\infty}\frac{a_k\varphi_{2k+1}(0)}{\beta_k^2 - 9/4}\right] \tag{4.6}$$

$$A^{(2)} = A^{(1)} + B^{(1)} - B^{(2)} + \left(\tfrac{11}{6} - 2\ln 2\right)(D^{(1)} - D^{(2)}) - \frac{3}{2}\sum_{k=1}^{\infty}\frac{X_k}{\mu_k^2 + 9/4}$$

In the derivation of these expressions, use has been made of the values

$$\int_0^1 (1 - \xi^2)P'_{-1/2+\mu_k i}(\xi)\,d\xi = -\frac{P''_{-1/2+\mu_k i}(0)}{\mu_k^2 + 9/4}$$
$$\int_0^1 (1 - \xi^2)P_{-1/2+\mu_k i}(\xi)\varphi_{2p+1}(\xi)\,d\xi = -\frac{P''_{-1/2+\mu_k i}(0)\,\varphi_{2p+1}(0)}{\mu_k^2 + (2p + 3/2)^2} \tag{4.7}$$

where

$$\varphi_{2p+1}(0) = (-1)^p\frac{(2p+1)!!}{(2p)!!} = (-1)^p\frac{1\cdot 3\cdot 5\cdots(2p+1)}{2\cdot 4\cdot 6\cdots 2p}, \qquad \varphi'_{2p+1}(0) = 0 \tag{4.8}$$

The unknown coefficients X_k and Y_k occurring in the Expressions (4.5) and (4.6) are determined by two infinite systems of linear equations

$$Y_p = \sum_{k=1}^{\infty} a_{pk}X_k + M_p, \qquad X_p = \sum_{k=1}^{\infty} b_{pk}Y_k + N_p \qquad (p = 1, 2, 3, \ldots) \tag{4.9}$$

where

$$a_{pk} = \frac{\beta_p^2 - {}^9/_4}{m\beta_p(\coth\beta_p t_1 + \coth\beta_p t_2)} \frac{1}{\mu_k^2 + \beta_p^2}$$

$$b_{pk} = -\frac{2m\mu_p^2}{t_1(\mu_p^2 + {}^9/_4)} \frac{P''_{-1/2+\mu_p i}(0)}{P'_{-1/2+\mu_p i}(0)} \frac{(4k+3)\varphi^2_{2k+1}(0)}{(2k+1)(2k+2)(\beta_k^2 + \mu_p^2)} \tag{4.10}$$

$$M_p = \frac{\beta_p^2 - {}^9/_4}{m\beta_p(\coth\beta_p t_1 + \coth\beta_p t_2)} \left[-\frac{D^{(1)} - D^{(2)}}{4p^2 + 6p} + \right.$$

$$\left. + \frac{(2p+1)(2p+2)}{4p(2p+3)\varphi_{2p+1}(0)} \left(\frac{a_p e^{-3t_1/2}}{\sinh\beta_p t_1} + \frac{b_p e^{3t_2/2}}{\sinh\beta_p t_2} \right) \right] \tag{4.11}$$

$$N_p = -\frac{2\mu_p^2}{t_1(\mu_p^2 + {}^9/_4)} \frac{P''_{-1/2+\mu_p i}(0)}{P'_{-1/2+\mu_p i}(0)} \left[(-1)^{p+1} e^{-3t_1/2} \sum_{k=1}^{\infty} \frac{a_k \varphi_{2k+1}(0)}{\beta_k^2 + \mu_p^2} - \right.$$

$$\left. \frac{e^{3t_2}(c_0 - 2b_0) - c_0 + 2(-1)^p a_0 e^{-3t_1/2}}{2(\mu_p^2 + {}^9/_4)} - \frac{d_p t_1}{2\mu_p}\left(\mu_p^2 + \frac{9}{4}\right) \right] \tag{4.12}$$

5. We will prove that the system (4.9) is completely regular. Making use of the Expressions (4.10), we will have

$$\sum_{k=1}^{\infty} |a_{pk}| = \frac{\beta_p^2 - {}^9/_4}{m\beta_p(\coth\beta_p t_1 + \coth\beta_p t_2)} \sum_{k=1}^{\infty} \frac{1}{\mu_k^2 + \beta_p^2} = \tag{5.1}$$

$$= \frac{t_1}{2m} \frac{\beta_p^2 - {}^9/_4}{\beta_p^2} \frac{\coth\beta_p t_1 - 1/\beta_p t_1}{\coth\beta_p t_1 + \coth\beta_p t_2} < \frac{t_1}{4m}$$

$$\sum_{k=1}^{\infty} |b_{pk}| = -\frac{2m}{t_1} \frac{\mu_p^2}{\mu_p^2 + {}^9/_4} \frac{P''_{-1/2+\mu_p i}(0)}{P'_{-1/2+\mu_p i}(0)} \sum_{k=1}^{\infty} \frac{(4k+3)\varphi^2_{2k+1}(0)}{(2k+1)(2k+2)(\beta_k^2 + \mu_p^2)} \tag{5.2}$$

For the summation of the series in the right-hand side of Expression (5.2), use can be made of the expansions of functions in series of Legendre functions.

Let the functions $f(\xi)$ and $\psi(\xi)$ belong to the class $L^2[0, 1]$ and be representable in the form of the series

$$f(\xi) = \sum_{k=0}^{\infty} a_k P_{2k+1}(\xi), \qquad \psi(\xi) = \sum_{k=0}^{\infty} b_k P_{2k+1}(\xi) \tag{5.3}$$

where the functions $P_{2k+1}(\xi)$ are orthogonal in the interval (0, 1) and satisfy the equalities

$$\int_0^1 P_{2k+1}(\xi)\,P_{2p+1}(\xi)\,d\xi = \begin{cases} 0 & (k \neq p) \\ \dfrac{1}{4k+3} & (k = p) \end{cases} \tag{5.4}$$

Since the series (5.3) are absolutely convergent, multiplying them together and integrating the resulting expressions over the interval (0, 1) leads to

$$\int_0^1 \psi(\xi)\, f(\xi)\, d\xi = \sum_{k=0}^{\infty} \frac{a_k b_k}{4k+3} \tag{5.5}$$

whereby use has been made of the equalities (5.4). The equality (5.5) can be called the Parseval equation for the expansions (5.3). Assuming now that

$$f(\xi) = \frac{P_{-1/2+\mu_p i}(\xi)}{P_{-1/2+\mu_p i}(0)}, \qquad \psi(\xi) = 1 \tag{5.6}$$

we will have

$$a_k = \frac{(4k+3)\,\varphi_{2k+1}(0)}{\mu_p^2 + (2k+3/2)^2}, \qquad b_k = \frac{(4k+3)\,\varphi_{2k+1}(0)}{(2k+1)(2k+2)} \tag{5.7}$$

Here use has been made of the integrals

$$\int_0^1 P_{2k+1}(\xi)\,d\xi = \frac{(\xi^2-1)\,P'_{2k+1}(\xi)}{(2k+1)(2k+2)}\Bigg|_0^1 = \frac{\varphi_{2k+1}(0)}{(2k+1)(2k+2)}$$

$$\int_0^1 P_{-1/2+\mu_p i}(\xi)\,P_{2k+1}(\xi)\,d\xi = -\frac{P_{-1/2+\mu_p i}(0)\,\varphi_{2k+1}(0)}{\mu_p^2 + (2k+3/2)^2} \tag{5.8}$$

Substituting the values (5.7) into (5.5), we obtain

$$\sum_{k=0}^{\infty} \frac{(4k+3)\,\varphi^2_{2k+1}(0)}{(2k+1)(2k+2)\,[\mu_p^2 + (2k+3/2)^2]} = -\frac{P'_{-1/2+\mu_p i}(0)}{(\mu_p^2 + 1/4)\,P_{-1/2+\mu_p i}(0)} \tag{5.9}$$

In the determination of this equality, use has been made of the integral

$$\int_0^1 P_{-1/2+\mu_p i}(\xi)\,d\xi = -\frac{(\xi^2-1)\,P'_{-1/2+\mu_p i}(\xi)}{\mu_p^2 + 1/4}\Bigg|_0^1 = -\frac{P'_{-1/2+\mu_p i}(0)}{\mu_p^2 + 1/4} \tag{5.10}$$

From the equation

$$(1-\xi^2) P''_{-1/2+\mu_p i}(\xi) - 2\xi P'_{-1/2+\mu_p i}(\xi) - (\mu_p^2 + 1/4) P_{-1/2+\mu_p i}(\xi) = 0 \quad (5.11)$$

when $\xi = 0$ we have

$$P''_{-1/2+\mu_p i}(0) = (\mu_p^2 + 1/4) P_{-1/2+\mu_p i}(0) \quad (5.12)$$

Substituting (5.12) into (5.9) we obtain

$$\sum_{k=1}^{\infty} \frac{(4k+3)\varphi^2_{2k+1}(0)}{(2k+1)(2k+2)[\mu_p^2 + (2k+3/2)^2]} = -\frac{P'_{-1/2+\mu_p i}(0)}{P''_{-1/2+\mu_p i}(0)} - \frac{3}{2(\mu_p^2 + 9/4)} \quad (5.13)$$

As a consequence of the relation (5.13) it is possible to assert that the expansion of the function $P'_{-1/2+\mu_p i}(\xi)$ in terms of the functions $\{\varphi_{2k+1}(\xi)\}$ will have the form

$$\frac{P'_{-1/2+\mu_p i}(\xi)}{P''_{-1/2+\mu_p i}(0)} = -\frac{3}{2(\mu_p^2 + 9/4)} - \sum_{k=1}^{\infty} \frac{(4k+3)\varphi_{2k+1}(0)\varphi_{2k+1}(\xi)}{(2k+1)(2k+2)[\mu_p^2 + (2k+3/2)^2]} \quad (5.14)$$

which remains valid at the boundary point $\xi = 0$.

As a result of (5.13), the equality (5.2) assumes the form

$$\sum_{k=1}^{\infty} |b_{pk}| = \frac{2m}{t_1} \frac{\mu_p^2}{\mu_p^2 + 9/4} \left[1 + \frac{3}{2(\mu_p^2 + 9/4)} \frac{P''_{-1/2+\mu_p i}(0)}{P'_{-1/2+\mu_p i}(0)}\right] \quad (5.15)$$

Since the right-hand side of (5.13) is positive, we have the bounds

$$-1 < \frac{3}{2(\mu_p^2 + 9/4)} \frac{P''_{-1/2+\mu_p i}(0)}{P'_{-1/2+\mu_p i}(0)} < 0 \quad (5.16)$$

By taking (5.16) into consideration, it follows from (5.15) that

$$\sum_{k=1}^{\infty} |b_{pk}| < \frac{2m}{t_1} \quad (5.17)$$

If the constant m is chosen from the equality

$$\frac{t_1}{4m} = \frac{2m}{t_1}, \qquad m = \frac{t_1\sqrt{2}}{4}$$

then, for the sums of the moduli of the coefficients of the unknowns of the systems (4.9), we will have the following estimates:

$$\sum_{k=1}^{\infty} |a_{pk}| < \frac{\sqrt{2}}{2}, \qquad \sum_{k=1}^{\infty} |b_{pk}| < \frac{\sqrt{2}}{2} \tag{5.18}$$

It is easy to see that the free terms (4.11) and (4.12) of the systems (4.9) are bounded above and that they vanish as $0(1/p)$ as $p \to \infty$.

Thus the systems (4.9) have turned out to be completely regular and to have free terms which are bounded above and tend to zero. This fact offers the possibility of determining all the unknowns X_k and Y_k to a desired degree of accuracy [14]. Moreover, it is easy to demonstrate that the unknown coefficients X_k and Y_k tend to zero as $0(k^{-1} \ln k)$.

6. Substituting the values of the unknown constants found from (4.5) into (2.6) and (2.7) for the functions ψ_1 and ψ_2, we finally find the following expressions:

$$\Psi_1(t, \xi) = A^{(1)} + B^{(1)}e^{-3t} + D^{(1)}\left[\frac{1}{1+\xi} - \ln(1+\xi) - t\right] +$$
$$+ \sum_{k=1}^{\infty} \frac{e^{-3t/2}\varphi_{2k+1}(\xi)}{\beta_k^2 - 9/4}\left[\frac{mY_k(4k+3)\varphi_{2k+1}(0)}{(2k+1)(2k+2)} R_k^{(1)}(t_1+t) - a_k e^{-3t_1/2} R_k^{(1)}(t)\right] +$$
$$+ \sum_{k=1}^{\infty} \frac{X_k + \gamma_k c_k}{\mu_k} \frac{P'_{-1/2+\mu_k i}(\xi)}{P''_{-1/2+\mu_k i}(0)} T_k^{(1)}(t) \tag{6.1}$$

$$\Psi_2(t, \xi) = A^{(2)} + B^{(2)}e^{-3t} + D^{(2)}\left[\frac{1}{1+\xi} - \ln(1+\xi) - t\right] -$$
$$- \sum_{k=1}^{\infty} \frac{e^{-3t/2}\varphi_{2k+t}(\xi)}{\beta_k^2 - 9/4}\left[\frac{mY_k(4k+3)\varphi_{2k+1}(0)}{(2k+1)(2k+2)} R_k^{(2)}(t-t_2) - b_k e^{3t_2/2} R_k^{(2)}(t)\right] +$$
$$+ \sum_{k=1}^{\infty} c_k \frac{P'_{-1/2+\gamma_k i}(\xi)}{P''_{-1/2+\gamma_k i}(0)} T_k^{(2)}(t) \tag{6.2}$$

In Equations (6.1) and (6.2) the notation

$$R_k^{(s)}(t) = \frac{3}{2}\frac{\sinh\beta_k t}{\sinh\beta_k t_s} + \beta_k \frac{\cosh\beta_k t}{\sinh\beta_k t_s} \qquad (s = 1, 2) \tag{6.3}$$

has been introduced.

The series which appear in the Formulas (6.1) and (6.2) are convergent. From the estimates for the unknown coefficients X_k and Y_k, it follows that the first derivatives of these series will be everywhere convergent except at the point ($t = 0$, $\xi = 0$), i.e. the shear stress at all points of the axial cross-section of the semisphere will be finite except at the point (0, 0) where there will be a stress concentration when the face is rigidly clamped.

Thus, by making use of the Formulas (1.5), (6.1) and (6.2), it is possible to determine the stresses $\tau_{t\varphi}$, $\tau_{\xi\varphi}$ and the displacement v at an arbitrary point of the axial section of the semisphere.

BIBLIOGRAPHY

1. Föppl, A., *Über die Torsion von runden Stäben mit veränderlichen Durchmesser.* Sitzungsber. Bayer. Akad. Wiss. München, 1905.

2. Lokshin, A.Sh., O kruchenii tela vrashcheniia (On the torsion of bodies of revolution). *Izv. Ekaterinoslav. gorn. in-ta* Vol. 11, No. 1, 1923.

3. Panarin, N.Ia., K raschetu usechennogo pologo konusa pri kruchenii (On the calculation of a truncated hollow cone subjected to torsion). *Izv. NIIG* Vol. 20, 1937.

4. Snell, C., The twisted sphere. *Matematica* Vol. 4, No. 8, 1957.

5. Huber, A., The elastic sphere under concentrated torques. *Quart. Appl. Math.* Vol. 13, 1955.

6. Solianik-Krassa, K.V., *Kruchenie valov peremennogo secheniia (Torsion of Shafts of Variable Section).* Gostekhizdat, 1949.

7. Abramian, B.L., O kruchenii tela vrashcheniia osesimmetrichnoi nagruzkoi (On the torsion of a solid of revolution under axisymmetrical loading). *PMM* Vol. 24, No. 6, 1960.

8. Das, S.Ch., On the stresses in a composite truncated cone due to shearing stresses on the curved surface. *Indian J. Theoret. Phys.* 4, No. 4, pp. 89-92, 1956.

9. Das, S.Ch., On the stresses in twisted composite spheres and spheroids. *Canad. J. Phys.* 35, No. 7, pp. 811-817, 1957.

10. Abramian, B.L. and Gulkanian, N.O., Kruchenie poloi dvukhsloinoi polusfery (The torsion of a hollow semisphere of two layers). *Izv. AN Arm. SSR, ser. fiz.-matem. nauk* Vol. 14, No. 6, 1961.

11. Hobson, E.W., *Teoriia sfericheskikh i ellipsoidal'nykh funktsii (Theory of Spherical and Ellipsoidal Harmonics).* IIL, 1952.

12. Ryzhik, I.M. and Gradshtein, I.S., *Tablitsy integralov, summ, riadov i proizvedenii (Tables of Integrals, Sums, Series and Products)*. GITTL, 1951.

13. Babloian, A.A., Ploskaia zadacha teorii uprugosti dlia kol'tsevogo secheniia (The plane problem in the theory of elasticity for an annular section). *Izv. AN Arm. SSR, ser. fiz.-matem. nauk* Vol. 14, No. 6, 1961.

14. Kantorovich, L.V. and Krylov, V.I., *Priblizhennye metody vysshego analiza (Approximate Methods of Higher Analysis)*. Gostekhizdat, 1950.

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